Knot theory in handlebodies

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Abstract

We consider oriented knots and links in a handlebody of genus g through appropriate braid representatives in S^3, which are elements of the braid groups $B_{g,n}$. We prove a geometric version of the Markov theorem for braid equivalence in the handlebody, which is based on the L-moves. Using this we then prove two algebraic versions of the Markov theorem. The first one uses the L-moves. The second one uses the Markov moves and conjugation in the groups $B_{g,n}$. We show that not all conjugations correspond to isotopies.

1 Introduction

A natural generalization of the classical knot theory in S^3 considers knots and links in more general 3-manifolds. While topological quantum field theories provide an approach to invariants of links in closed (i.e. compact without boundary) 3-manifolds, bounded 3-manifolds are also of interest, since –for once– they give rise to closed, connected, orientable 3-manifolds. In particular, we have on the one hand handlebodies, which give rise to 3-manifolds via the Heegaard decomposition, and on the other hand knot complements, which give rise to 3-manifolds via the surgery technique. In [9] knots and links in knot complements and 3-manifolds are studied via braids. Here we study knots and links in handlebodies. The special case of the solid torus is the only bounded manifold common in both categories, and its knot theory has been studied quite extensively from various viewpoints (see [19], [5], [6], [10, 3, 11], [8], [2], [4]). Various aspects of the knot theory of a handlebody have been studied in [15], [16], [18], [13], [20], [14]. Let now H_g denote a handlebody of genus g. A handlebody of genus g is usually defined as (*a closed disc* $\setminus$ $\{$*g open discs*$\}$) $\times I$, where I is the unit interval. See Fig. 1.

Equivalently, H_g can be defined as ($S^3 \setminus$ *an open tubular neighbourhood of* I_g), where I_g denotes the pointwise fixed identity braid on g indefinitely extended strands, all meeting at the point at infinity, see Fig. 2. Thus H_g *may be represented in* S^3 by the braid I_g. Now let L be an oriented link in H_g. Then L will avoid the g hollow tubes of H_g, and also it will not pass beyond the boundary

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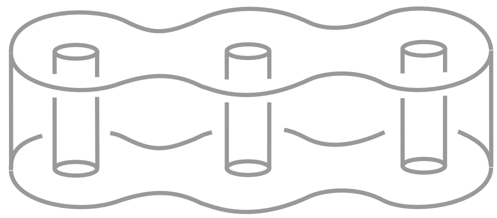

Figure 1: **A handlebody of genus** 3

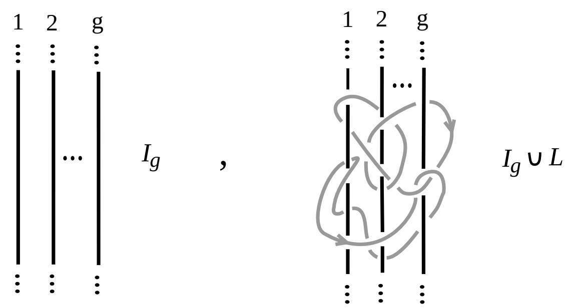

Figure 2: Representation of H_g - a mixed link

of H_g from either end. Equivalently, the link L in H_g *may be represented unambiguously* by the mixed (g,g)-tangle $I_g \bigcup L$ in S^3, which by abuse of language we shall call *mixed link* (see Fig. 2). The subbraid I_g shall be called *the fixed part* and L *the moving part* of the mixed link. A *mixed link diagram* $I_g \bigcup \tilde{L}$ is then a diagram of $I_g \bigcup L$ projected on the plane of I_g, which is equipped with the top-to-bottom direction. Note that, if we remove I_g from a mixed link we are left with an oriented link in S^3.

In this paper we study isotopy of oriented links and equivalence of braids in H_g via their mixed link and mixed braid representatives in S^3. The paper is organised as follows. In Section 2 we study knot isotopy in handlebodies combinatorially, we establish the notion of a braid in H_g, and we prove that every oriented link in H_g can be braided. In Section 3 we prove a geometric version of the Markov theorem for oriented links in H_g (Theorem 3) using the so-called L-moves (Definition 6) and the Relative Version of Markov theorem. In Section 4 we define the algebraic structures of braids in S^3 that represent braids in H_g and we prove two algebraic versions of the Markov theorem for handlebodies. The first one (Theorem 4) uses the L-moves. The second one (Theorem 5) uses a presentation of the groups $B_{g,n}$, and its formulation resembles the classical Markov theorem for S^3. Only, here the Markov move (the one that introduces a twist) has to take place anywhere on the right of the braid. Also, as we prove, not all conjugations in the groups $B_{g,n}$ induce isotopy in the handlebody. This disproves a conjecture of A. Sossinsky, [18]. In Section 5 we discuss which conjugations are allowed (Theorem 6). Finally, in Section 6 we discuss what kind of maps should be defined on appropriate quotient algebras in order to replace the notion of a Markov trace.

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2 Knots and braids in H_g

Throughout the paper the handlebody H_g will be represented in S^3 by the braid I_g, as defined in Section 1, and a link L in H_g will be represented by the mixed link $I_g \bigcup L$ in S^3. The set-up is similar to the one of [9], and we will refer to [9] for the proofs of results needed here and already established there. Otherwise, we have tried to present our results in a self-contained manner. All links will be assumed oriented and all diagrams piecewise linear (PL). Whenever we say 'knots' we mean 'knots and links'. Finally, we will be thinking in terms of diagrams for both knots and braids.

Definition 1. Two oriented links L_1, L_2 in H_g are *isotopic* if and only if there is an ambient isotopy of $(S^3 \setminus I_g,\ L_1) \longrightarrow (S^3 \setminus I_g,\ L_2)$ taking L_1 to L_2. Equivalently, L_1 and L_2 are isotopic in H_g if and only if the mixed links $I_g \bigcup L_1$ and $I_g \bigcup L_2$ are isotopic in S^3 by an ambient isotopy which keeps I_g pointwise fixed.

In the PL category ambient isotopy is realized through a finite sequence of the so-called Δ-*moves* in three-space.

Definition 2. A Δ-move on a link L in H_g is an elementary combinatorial isotopy move (and its inverse), realized by replacing an arc of L by two other arcs respecting orientation, and such that all three arcs span a triangle in space, the spanning surface of which does not intersect any other arcs of L. On the level of the mixed link $I_g \bigcup L$ in S^3, a Δ-move applies only on the moving part. A Δ-move on a mixed link diagram $I_g \bigcup \tilde{L}$ is the regular projection of a Δ-move on the plane of I_g.

Definition 3. A *non-critical* Δ-*move* on a link L in H_g is a Δ-move, such that on its regular projection on the plane of the subbraid I_g nothing critical occurs if we remove the subbraid I_g. Consequently, on the level of the mixed link diagram $I_g \bigcup \tilde{L}$, a non-critical Δ-move will be a Δ-move on $\tilde{L}$, whose spanning triangle either does not meet any other arcs on the plane of projection (and so it is a *planar* Δ-*move* in the classical set-up) or it meets parts of the fixed

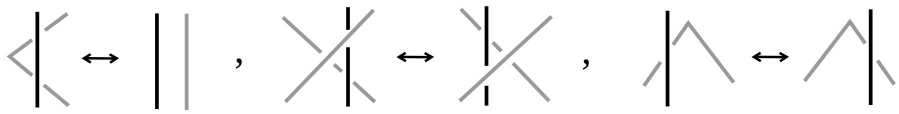

Figure 3: Mixed isotopy moves

subbraid I_g. These last possibilities shall be called *mixed isotopy moves*, see Fig. 3.

Reidemeister [17] (and Alexander, Briggs [1]) proved that a Δ-move on a link diagram in S^3 can break into a finite sequence of the three local Δ-moves known as 'Reidemeister moves', and of planar Δ-moves, i.e. moves, whose spanning triangle does not meet any other arcs on the projection plane, with their obvious symmetries and choices of orientation. From the above we deduce that knot isotopy in H_g is realized combinatorially through the following (compare with the relative version of Reidemeister theorem, Theorem 5.2 of [9]):

Theorem 1 (Reidemeister theorem for H_g). *Two oriented links in H_g are isotopic if and only if any two corresponding mixed link diagrams in S^3 differ by a finite sequence of planar Δ-moves, the three Reidemeister moves and the mixed isotopy moves (with their obvious different choices of orientation, crossings and direction), all of which apply only on the moving parts of the diagrams.*

Assuming that the strands of I_g are oriented downwards, we can now define:

Definition 4. A *geometric mixed braid* on n strands, denoted $I_g \bigcup B$, is an element of the classical braid group B_{g+n}, consisting of two disjoint sets of strands, one of which is the identity braid I_g, whilst the other set of strands has labels 'u' or 'o' (for 'under' or 'over') attached to each pair of corresponding endpoints (see Fig. 4). For the two sets of strands we use the terms *I_g-part* for the identity subbraid and *B_n-part* for the labelled subbraid B. The reason for choosing this notation will become clear soon. A *diagram of a geometric mixed braid* is a braid diagram in the usual sense, projected on the plane of I_g.

Fig. 4 illustrates an abstract geometric mixed braid enclosed in a 'box', as well as an example in B_6. Note that the set of geometric mixed braids on n strands does not form a group, as composition may not be well-defined. Geometric mixed braids in H_g may be visualized as having endpoints on three different parallel planes, parallel to the plane of the paper, such that the subbraid I_g lies on the middle one, the endpoints labelled 'o' lie on the front plane (the one closest to the reader), and the endpoints labelled 'u' lie on the back plane (the furthermost from the reader).

We obtain knots from braids via the well-known closing operation adapted to our situation. So, we have:

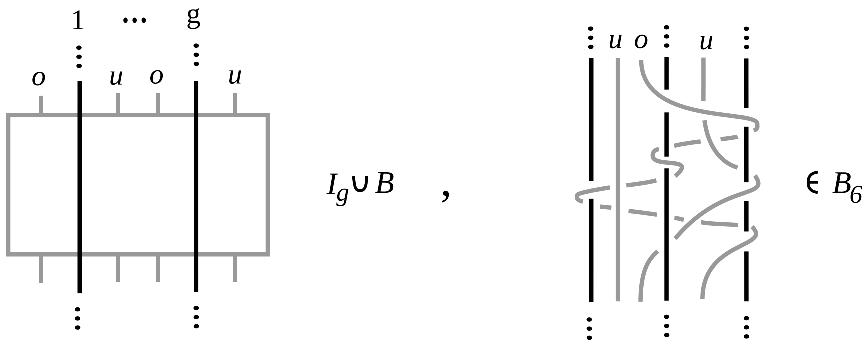

Figure 4: Geometric mixed braids

Definition 5. The *closure* $\mathcal{C}(I_g \bigcup B)$ of a geometric mixed braid $I_g \bigcup B$ is an operation that results in an oriented mixed link, and it is realized by joining each pair of corresponding (slightly bent) endpoints *of the B_n-part* by a vertical segment, either *over* or *under* the rest of the braid, according to the label attached to these endpoints (see Fig. 5 for an example).

Note that the strands of I_g do not participate in the closure operation, that's why they are assumed to be infinitely extensible. Besides, the labelling 'u' or 'o' for corresponding endpoints in Definition 4 is precisely an instruction on how to perform the closure. Different choices of labels will yield in general non-isotopic links in H_g, as the example in Fig. 6 illustrates. We return to this example in the discussion before Fig. 19.

Remark 1. Let M denote the complement of the g-unlink or a connected sum of g lens spaces of type $L(p, 1)$. Then braids in M can be also represented in S^3 by unlabelled geometric mixed braids with I_g as a fixed subbraid (cf. [9]). Note that in both H_g and M, if we remove I_g from a mixed braid, we are left with a braid in S^3. This will be a labelled braid in the case of H_g. But this is equivalent to the familiar unlabelled picture of a classical braid, since a closing arc labelled 'o' can slide freely over to the side and then to the back of the braid, thus aquiring the label 'u' (see Fig. 7). This isotopy is the reason that mixed braids in M are not labelled, since in the set-up of [9] I_g participates also in the closure of the braid (contrary to H_g).

Conversely to the closure of braids, mixed links may be braided, so that if we start with a mixed link, do braiding and then take closure, we obtain a mixed link isotopic to the original one. Indeed, we have:

Theorem 2 (Alexander theorem for H_g). *An oriented mixed link $I_g \bigcup L$ in H_g may be braided to a geometric mixed braid, the closure of which is isotopic to $I_g \bigcup L$.*

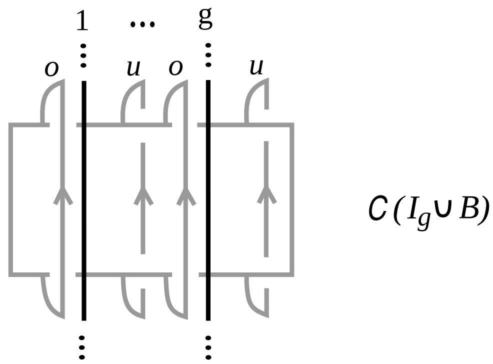

Figure 5: Closure of a geometric mixed braid

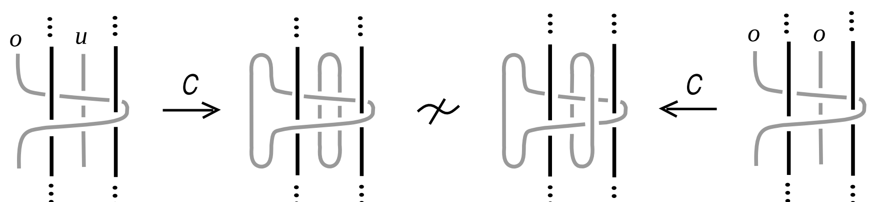

Figure 6: Different labels yield non–isotopic links

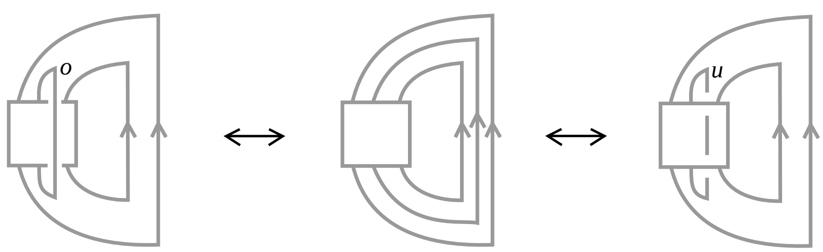

Figure 7: The 'under – over' interchange

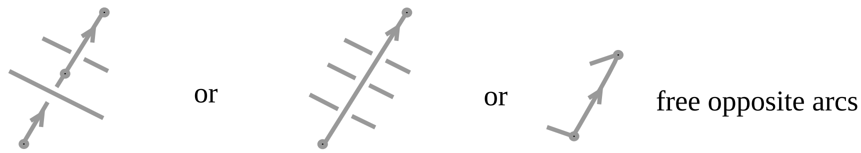

Figure 8:

Proof. We apply the braiding algorithm of [9] on a diagram $I_g \bigcup \tilde{L}$ of the PL mixed link $I_g \bigcup L$. By general position $I_g \bigcup \tilde{L}$ contains no horizontal arcs with respect to the height function. The idea of the braiding is on the one hand to keep the arcs of the diagram that are oriented downwards with respect to the height function, and on the other hand to eliminate the ones that go upwards and produce instead braid strands. We call these arcs *opposite arcs*. Now, the point is that the subbraid I_g will not be touched by the algorithm, so the opposite arcs will be arcs of the link L. The elimination of the opposite arcs is based on the following: If we run along an opposite arc we are likely to meet a succession of overcrossings and undercrossings. We subdivide (marking with points) every opposite arc into smaller – if necessary – pieces, each containing crossings of only *one* type; i.e. we may have:

We call the resulting pieces *up-arcs*, and we label every up-arc with an *'o'* resp. *'u'* according as it is the *over* resp. *under* arc of a crossing (or some crossings). If it is a *free up-arc* (one that contains no crossings), then we have a choice whether to label it *'o'* or *'u'*. The idea is to eliminate the opposite arc by eliminating its up-arcs one by one and create instead a pair of braid strands for each up-arc.

Let now P be the top vertex of the up-arc QP (see Fig. 9). Associated to QP is the *sliding triangle* $T(P)$, which is a special case of a triangle needed for a Δ–move; it is right-angled with hypotenuse QP and with the right angle lying below the up-arc. Note that, if QP is itself vertical, then $T(P)$ degenerates into the arc QP. We say that a sliding triangle is of type *over* or *under* according to the label of the up-arc it is associated with. (This implies that there may be triangles of the same type lying one on top of the other.)

The germ of our braiding process is this. Suppose for definiteness that QP is of type *over*. Then we cut QP at P and we pull the two ends, the top upwards and the lower downwards, and both *over* the rest of the diagram, so as to create a pair of corresponding strands of the anticipated braid (see Fig. 9). Finally, we perform a Δ-move across the sliding triangle $T(P)$. By general position the resulting diagram will be regular and QQ' may be assumed to slope slightly downwards. If QP were *under* then the pulling of the two ends would be *under* the rest of the diagram. Note that the effect of these two operations has been to replace the up-arc QP by three arcs none of which is an up-arc, and the two of them being corresponding braid strands. Therefore we now have fewer up-arcs.

For each up-arc that we eliminate, we label the corresponding end strands

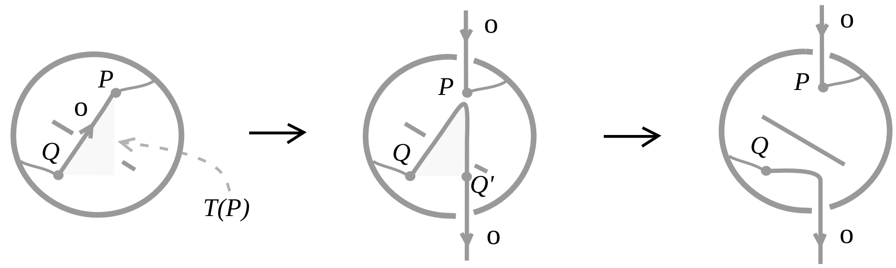

Figure 9: The germ of the braiding

'o' or 'u' according to the label of their up-arc. (As already noted, in [9] this labelling was not needed.) After eliminating all up-arcs we obtain a geometric mixed braid, denoted $\mathcal{B}(I_g \bigcup L)$, the closure of which is obviously isotopic to $I_g \bigcup L$. Indeed, from Definition 5, the closing arc of two corresponding endpoints of the braid is precisely a stretched version of the initial up-arc, since it bears the same label. □

The proof of Theorem 2 is analogous to the one in Section 3 of [9]. We have repeated it here for the sake of completion.

3 Geometric Markov theorem for H_g

As in classical knot theory, the next consideration is how to characterize geometric mixed braids that induce via closure isotopic links in H_g. For this purpose we need to recall the L-moves between braids. These were introduced in [9], and they generalize geometrically the Markov moves in the following sense. An L_o-*move* resp. L_u-*move* on a braid consists of cutting an arc open and splicing into the broken strand new arcs to the top and bottom, both *over* resp. *under* the rest of the braid (see Fig. 10 for the case of H_g). As remarked in [9], using a small braid isotopy, a braid L-move can be equivalently seen with a crossing (positive or negative) formed (see Fig. 11 for H_g). Therefore, a geometric Markov move in a braid, that introduces a crossing in the bottom right position, is a special case of an L-move. L-moves and braid isotopy generate an equivalence relation on braids called L-*equivalence*. It was shown in [9] that L-equivalent classes of braids are in bijective correspondence with isotopy classes of oriented links in S^3, the bijection being induced by 'closing' the braid. Modified slightly, L-moves in a handlebody are defined as follows.

Definition 6 (Geometric L-moves in H_g). Let $I_g \bigcup B$ be a braid in H_g and P a point of an arc of the subbraid B, such that P is not vertically aligned with any crossing. Doing a *geometric L-move* at P means to perform the following operation: cut the arc at P, bend the two resulting smaller arcs slightly apart

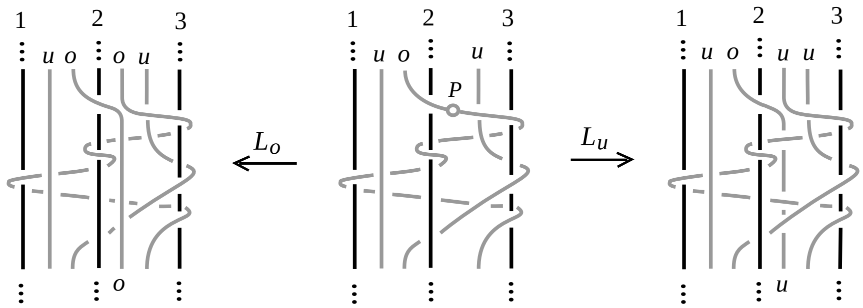

Figure 10: The two types of L–moves in H_g

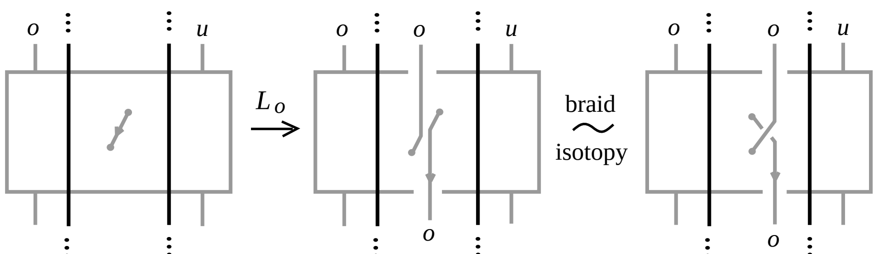

Figure 11: An L–move introduces a crossing

by a small isotopy and stretch them vertically, the upper downwards and the lower upwards, and both *over* or *under* all other arcs of the diagram, so as to introduce two new corresponding strands with endpoints on the vertical line of P, labelled 'o' or 'u' according to the stretching. Stretching the new strands over will give rise to a *geometric L_o-move* and under to an *geometric L_u-move*. Undoing an L-move is defined to be the reverse operation. Also in this set-up, two geometric mixed braids in H_g that differ by an L-move shall be called *L-equivalent*.

Fig. 10 illustrates an example of a geometric L_o-move and a geometric L_u-move at the same point of a geometric mixed braid, whilst Fig. 11 illustrates an abstract geometric L_o-move and the crossing it introduces in the braid box.

Remark 2. L-equivalent geometric mixed braids have isotopic closures, since the labels we give to the new endpoints after performing an L-move on a mixed braid agree with the type of the L-move. So closure is compatible with the L-move, and it corresponds to introducing a twist in the mixed link.

We are now in a position to state the following.

Theorem 3 (Geometric version of Markov theorem for H_g). *Two oriented links in H_g are isotopic iff any two corresponding geometric mixed braids differ by a finite sequence of L-moves and isotopies of geometric mixed braids.*

Proof. Let $\mathcal{B}$ denote the braiding map of Theorem 2, let $I_g \bigcup \tilde{L}$ be a mixed link diagram in H_g, and let $I_g \bigcup B = \mathcal{B}(I_g \bigcup \tilde{L})$. By Theorem 2, $\mathcal{C} \circ \mathcal{B}(I_g \bigcup \tilde{L})$ is isotopic to $I_g \bigcup \tilde{L}$. Further, $\mathcal{B} \circ \mathcal{C}(I_g \bigcup B) = I_g \bigcup B$. This follows from Definition 5 and from the fact that if we braid the closing arcs of a mixed braid, $I_g \bigcup B$ say, each closing arc will give rise to one pair of corresponding strands, so we obtain again the braid $I_g \bigcup B$.

We now consider the liftings $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ of the maps $\mathcal{B}$ and $\mathcal{C}$ on isotopy classes of link diagrams and on L-equivalent classes of geometric mixed braids respectively. We will show that $\tilde{\mathcal{C}}$ is a bijection with inverse $\tilde{\mathcal{B}}$. It follows from Remark 2 that $\tilde{\mathcal{C}}$ is well-defined. Thus, from the observations above, it only remains to show that $\tilde{\mathcal{B}}$ is also well-defined, that is to show that geometric mixed braids corresponding to isotopic mixed links are L-equivalent. For this we apply the:

Relative Version of Markov theorem (Theorem 4.7 of [9]) *Let L_1, L_2 be oriented link diagrams in S^3, both containing a common braided portion B. Suppose that there is an isotopy of L_1 to L_2 which finishes with a homeomorphism fixed on B. Suppose further that B_1 and B_2 are braids obtained from our braiding process applied to L_1 and L_2 respectively, both containing the common braided portion B. Then B_1 and B_2 are L–equivalent by moves that do not affect the braid B.*

Here I_g plays the role of the the common subbraid B, which, by Definition 3 and by Theorem 1, remains fixed throughout an isotopy of two mixed link diagrams $I_g \bigcup \tilde{L}_1$ and $I_g \bigcup \tilde{L}_2$. Further, the braiding $\mathcal{B}$ keeps I_g fixed in the corresponding geometric mixed braids, $I_g \bigcup B_1$ and $I_g \bigcup B_2$, say. Thus, the relative version of Markov theorem guarantees that $I_g \bigcup B_1$ and $I_g \bigcup B_2$ are L-equivalent by L-moves that do not affect I_g. But this is precisely the definition of L-moves in H_g (recall Definition 6). The only difference from S^3 is that here we attach labels to the corresponding strands of each L-move according to its type. In S^3 this was not needed. $\square$

4 Algebraic versions of Markov theorem

In order to construct invariants of knots in the handlebody using the braid approach we must translate Theorem 3 into algebra (see for example [7] for S^3 and [11] for the solid torus). For this we need first to introduce the braid groups $B_{g,n}$.

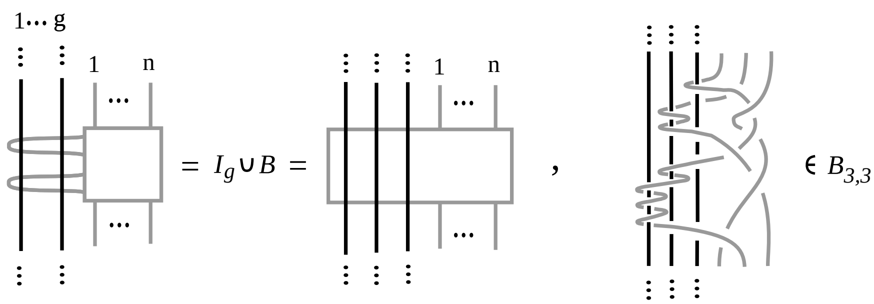

Figure 12: Algebraic mixed braids

Definition 7. An *algebraic mixed braid* on n strands is an element of the braid group B_{g+n} consisting of two disjoint sets of strands, such that *the first* g *strands* constitute the identity braid I_g. We denote algebraic mixed braids in the same way as the geometric mixed braids.

Fig. 12 suggests two ways for depicting abstractly algebraic mixed braids, and it gives a concrete example of an algebraic mixed braid on three strands.

We shall see that an algebraic mixed braid is a special case of a geometric mixed braid. Clearly, it is a special case of an *unlabelled* geometric mixed braid. We say that an algebraic mixed braid *is made geometric* if we attach arbitrary labels 'u' or 'o' at its corresponding endpoints. Note that this is an ambiguous process.

Definition 8. The *closure* of an algebraic mixed braid $I_g \bigcup B$, denoted $I_g \bigcup \widehat{B}$, is defined by joining each pair of the (slightly bent) corresponding endpoints of the B_n-part by a vertical segment (see left illustration of Fig. 13).

Remark 3. If we consider an algebraic mixed braid $I_g \bigcup B$ made geometric, its closure $\mathcal{C}(I_g \bigcup B)$ is isotopic to $I_g \bigcup \widehat{B}$, no matter what labels we used for the B_n-part, since the closing arcs can be stretched and can slide freely over to the right-hand side of the braid (see Fig. 13). *This shows that algebraic mixed braids are indeed special cases of geometric mixed braids, for which labels are superfluous.*

Conversely, geometric mixed braids can be made algebraic. Indeed, the operation 'closure' is an equivalence relation in the set of geometric mixed braids, and we have:

Lemma 1. *Every geometric mixed braid may be represented by an algebraic mixed braid with isotopic closure.*

Proof. Pull each pair of corresponding endpoints of the geometric mixed braid $I_g \bigcup B$ to the right side of I_g *over* or *under* the strands of I_g according to

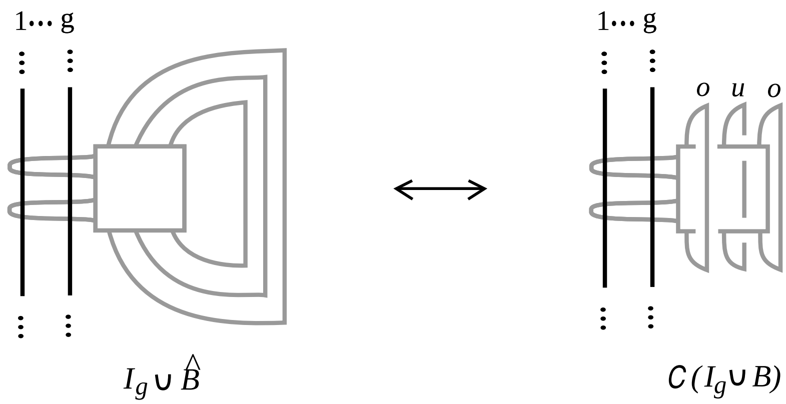

Figure 13: Closure of an algebraic mixed braid

its label, starting from the rightmost pair, and respecting the position of the endpoints. Schematically:

We thus obtain unambiguously an algebraic mixed braid. We denote this last step of the braiding algorithm by $\mathcal{A}$, and we say that through $\mathcal{A}$ a geometric mixed braid is made algebraic. Now, $\mathcal{C}(I_g \bigcup B)$ is isotopic to the closure of the algebraic mixed braid $\mathcal{A}(I_g \bigcup B)$. To see this we choose as labels of the algebraic mixed braid $\mathcal{A}(I_g \bigcup B)$ the initial labels of the geometric mixed braid $I_g \bigcup B$. Then the closures of the two geometric mixed braids are isotopic, and, by Remark 3 above, the assertion is proved. □

As an example, the algebraization of the two geometric braids of Fig. 6 are illustrated in Fig. 21.

The sets of algebraic mixed braids on n strands, denoted $B_{g,n}$, have been treated in [12]. It is shown there that these are the underlying braid structures for studying knots in a handlebody, in knot complements and in closed, connected, orientable 3-manifolds. Moreover, they form subgroups of the groups B_{g+n} with operation the usual concatenation, and with presentation:

$$B_{g,n} = \left\langle \begin{array}{c} a_1, \ldots, a_g, \\ \sigma_1, \ldots, \sigma_{n-1} \end{array} \middle| \begin{array}{l} \sigma_k \sigma_j = \sigma_j \sigma_k, \quad |k-j| > 1 \\ \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \quad 1 \leq k \leq n-1 \\ a_i \sigma_k = \sigma_k a_i, \quad k \geq 2, \ 1 \leq i \leq g, \\ a_i \sigma_1 a_i \sigma_1 = \sigma_1 a_i \sigma_1 a_i, \quad 1 \leq i \leq g \\ a_i (\sigma_1 a_r {\sigma_1}^{-1}) = (\sigma_1 a_r {\sigma_1}^{-1}) a_i, \quad r < i. \end{array} \right\rangle,$$

where the generators of $B_{g,n}$ may be represented geometrically by:

Let $B_{g,\infty} := \bigcup_{n=1}^{\infty} B_{g,n}$ denote the disjoint union of all braid groups $B_{g,n}$ (not the inductive limit). Proceeding towards the algebraization of Theorem 3 we define:

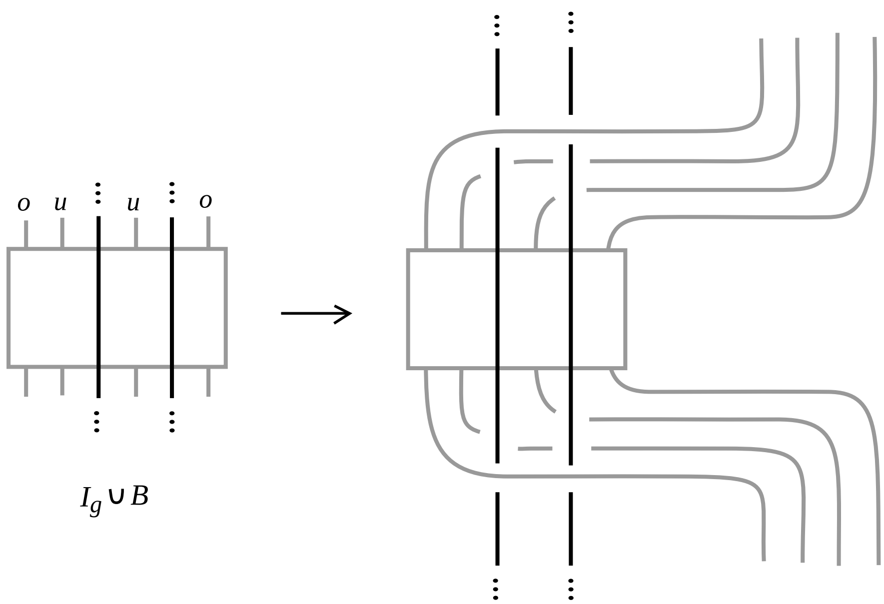

Figure 14: Algebraization of a geometric mixed braid

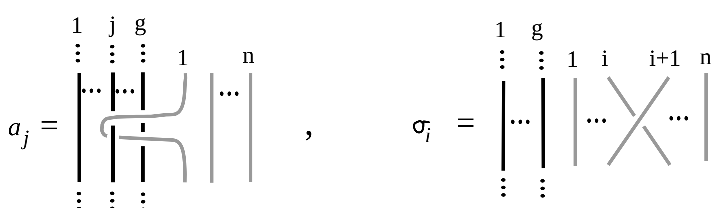

Figure 15: The generators of $B_{g,n}$

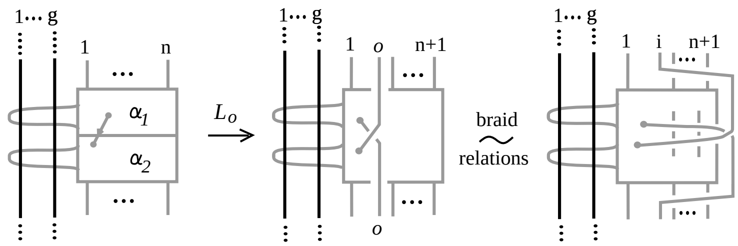

Figure 16: Algebraic expression of an algebraic L_o-move

Definition 9. An *algebraic L-move* is a geometric L-move between elements of $\bigcup_{n=1}^{\infty} B_{g,n}$, i.e. an L-move that preserves the group structure of the algebraic mixed braids.

It follows from Remark 3 that algebraic L-moves do not need the labels 'o' and 'u'. In some illustrations we keep the labels for the sake of clarity. An algebraic L-move in a braid $\alpha \in B_{g,n}$ has the following algebraic expressions, depending on its type. These are easily derived, as Fig. 16 shows.

$$L_o\text{-type:}\quad \alpha = \alpha_1\alpha_2 \ \sim\ \sigma_i^{-1}\ldots\sigma_n^{-1}\alpha_1\sigma_{i-1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_n^{\pm 1}\sigma_{n-1}\ldots\sigma_i\alpha_2\sigma_n\ldots\sigma_i,$$

$$L_u\text{-type:}\quad \alpha = \alpha_1\alpha_2 \ \sim\ \sigma_i\ldots\sigma_n\alpha_1\sigma_{i-1}\ldots\sigma_{n-1}\sigma_n^{\pm 1}\sigma_{n-1}^{-1}\ldots\sigma_i^{-1}\alpha_2\sigma_n^{-1}\ldots\sigma_i^{-1}.$$

Lemma 2. *Consider a geometric mixed braid containing a geometric L-move, which is made algebraic. Then the L-move is turned into an algebraic L-move.*

Proof. Since the type of a geometric L-move agrees with the label of its endpoints, by pulling the endpoints to the right the crossing of the L-move slides over by a braid isotopy. Schematically:
The case of a geometric L_u-move is completely analogous. Here the pulling takes place under the braid, so the crossing of the geometric L_u-move slides along to the right to form an algebraic L_u-move. □

Now we can state the following:

Theorem 4 (First algebraic version of Markov theorem for H_g). *Two oriented links in H_g are isotopic iff any two corresponding algebraic mixed braids differ by a finite sequence of algebraic L-moves and the braid relations in $\bigcup_{n=1}^{\infty} B_{g,n}$.*

Proof. It follows from Theorem 3 and Lemma 2. □

Remark 4. Theorem 4 is a rephrasing of Theorem 3 in an algebraic set-up. One could omit Theorem 3 and prove Theorem 4 directly using the Relative

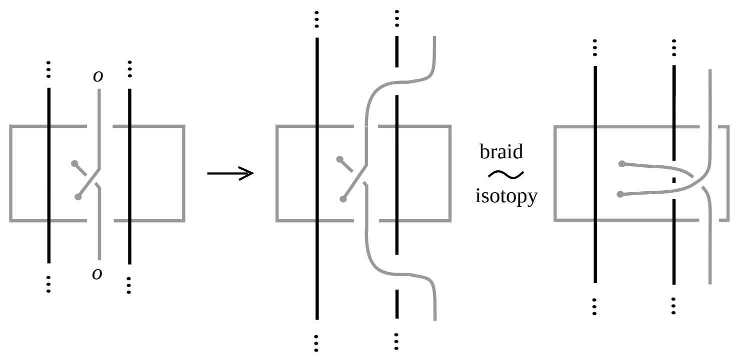

Figure 17: Sliding a geometric L_o-move to the right

Version of Markov theorem, after incorporating in the braiding algorithm $\mathcal{B}$ the last algebraization step $\mathcal{A}$. We decided to separate the two results, so as to stress the passage from the geometric to the algebraic set-up and the results that are valid in each one.

In order to look for Markov functionals on $B_{g,\infty}$, so as to construct link invariants in H_g, we further prove:

Theorem 5 (Second algebraic version of Markov theorem for H_g). *Two oriented links in H_g are isotopic iff any two corresponding algebraic mixed braids differ by a finite sequence of the following moves:*

1. *Markov move:* $\beta_1\beta_2 \sim \beta_1\sigma_n^{\pm 1}\beta_2, \quad \beta_1, \beta_2 \in B_{g,n}$
2. *Markov conjugation:* ${\sigma_i}^{-1}\beta\sigma_i \sim \beta, \quad \beta, \sigma_i \in B_{g,n}$

Proof. The two types of moves are illustrated in Figs. 18a and 18b. It is easy to see that both do not leave the isotopy class of the link. In fact, the first one is simply a special case of an algebraic L-move that takes place at the rightmost part of the algebraic mixed braid, whilst the second one clearly induces isotopy via closure, as defined in Definition 8. The other direction is shown by reducing to Theorem 4. Indeed, an algebraic L-move can be realized by a finite sequence of the above moves, as it follows clearly from the algebraic expressions of the two types of algebraic L-moves (recall Fig. 16). □

Algebraic mixed braids that are equivalent in the context of Theorem 4 or Theorem 5 shall be called *Markov equivalent.* A remark is now in order.

Remark 5. In the classical case in S^3 the braid move $\alpha_1\sigma_n^{\pm 1}\alpha_2 \sim \alpha_1\alpha_2$ is equivalent to the move

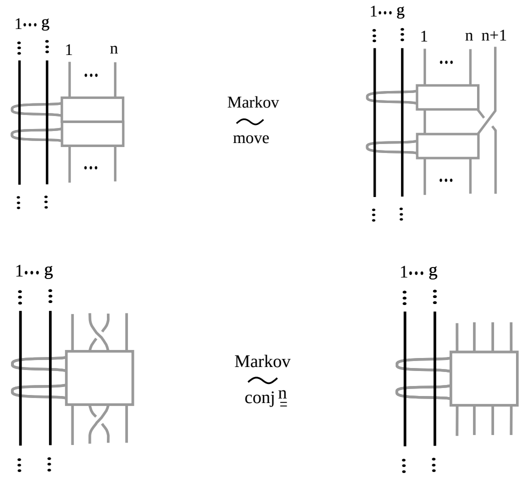

Figure 18: The Markov move in H_g and Markov conjugation in H_g

$$\alpha\sigma_n^{\pm 1} \sim \alpha,$$

where $\alpha = \alpha_1\alpha_2 \in B_n$. This is still true in the case of a solid torus (see [10], [4]). To see this think that the infinite strand of a solid torus may be closed at the point at infinity, so any loop can conjugate with no obstruction. In [4], Lemma 39, it is shown how to commute a loop from the bottom to the top of the braid without closing the infinite strand. But in a handlebody of genus greater than one this is not the case any more. Here the braid word α_2 may contain more than one of the g generators a_i of the braid group (recall Fig. 15). This is discussed in detail in the next section.

5 On hidden conjugations

There are two natural questions arising now:

(1) are there any 'hidden' conjugations involving the generators a_i, which preserve the isotopy class of the closure of a mixed braid (even though the strands of I_g do not participate in the closure)?

(2) if yes, are all conjugations 'allowed'?

Before answering we need to introduce another notion.

Definition 10. A *loop* in $B_{g,n}$ is a word of the form $b_i := a_i a_{i+1} \cdots a_g$ or its inverse, for $i < g$, and a *maximal loop* the word $b_1 := a_1 a_2 \cdots a_g$ or its inverse (see Fig. 19 for illustrations). A maximal loop shall be denoted by ω.

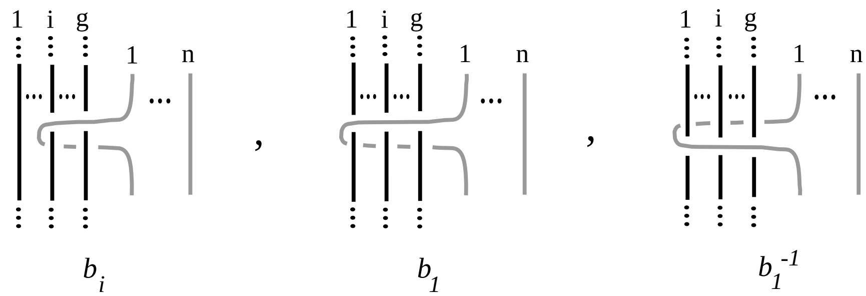

Figure 19: A loop and the two maximal loops

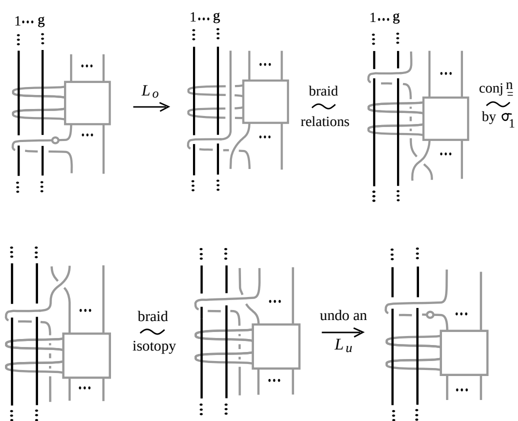

Figure 20: The proof of Lemma 3

The answer to the first question is positive. Indeed, we have:

Lemma 3. *Let $\alpha \in B_{g,n}$ be arbitrary and let ω be a maximal loop. Then the braids $\alpha\,\omega$ and $\omega\,\alpha$ are Markov equivalent.*

Proof. Fig. 20 demonstrates that, using algebraic L-moves and conjugation by a σ_1, the given algebraic mixed braids are Markov equivalent by Theorem 5. Thus their closures are isotopic. □

Remark 6. An alternative proof of Lemma 3 would be to take the closures of $\omega^{-1}\,\alpha\,\omega$ and α and to observe that a closing arc of $\omega^{-1}\,\alpha\,\omega$ can be dragged around to the left and all the way round to the position α.

The answer to the second question is negative, as the example below shows: The two algebraic mixed braids of Fig. 21 are the algebraizations of the geometric

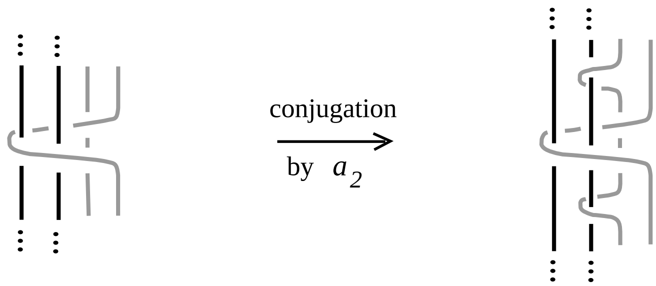

Figure 21: Two non–isotopic conjugate algebraic mixed braids

mixed braids of Fig. 6, thus their closures are the two non-isotopic mixed links of Fig. 6.

Now a third question arises:

(3) Can we list all the conjugations that are allowed with respect to isotopy?

In order to answer this question we give first another presentation of the braid group $B_{g,n}$ with the b_i's as generators. This presentation is easily derived from the one with the generators a_i given in the previous section using that $a_i = b_i {b_{i+1}}^{-1}$.

$$B_{g,n} = \left\langle \begin{array}{c} b_1, \ldots, b_g, \\ \sigma_1, \ldots, \sigma_{n-1} \end{array} \left| \begin{array}{l} \sigma_k \sigma_j = \sigma_j \sigma_k, \quad |k-j| > 1 \\ \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \quad 1 \leq k \leq n-1 \\ b_i \sigma_k = \sigma_k b_i, \quad k \geq 2, \ 1 \leq i \leq g \\ b_i \sigma_1 b_r \sigma_1 = \sigma_1 b_r \sigma_1 b_i, \quad r \leq i \end{array} \right. \right\rangle.$$

It is important to understand that *conjugation by some* b_i of an algebraic mixed braid *is equivalent to changing the labels* of some pair of corresponding endpoints in a related geometric mixed braid. If, for example, the two corresponding endpoints of the geometric mixed braid lie to the left of all strands of I_g, then by a braid isotopy part of which is absorbed inside the braid box, the geometric mixed braid can look like one of the middle pictures of Fig. 22. Thus, change of labels corresponds to conjugating the algebraization of the 'o' braid (resp. the 'u' braid) by the loop b_1 (resp. ${b_1}^{-1}$), as Fig. 22 demonstrates.

But these conjugations are allowed as we proved in Lemma 3. (To see the isotopy on the level of the geometric mixed braids look at the closing arc of the left middle braid of Fig. 22. This can pass from the 'u' position to the 'o' position without any obstruction from the braid.) Thus, change of labels in this case reflects isotopy between the closures of the two geometric mixed braids.

Let, now, the two corresponding endpoints of a geometric mixed braid lie between the ith and $(i+1)$st strand of I_g, for $i \neq 1$. Then, with similar reasoning

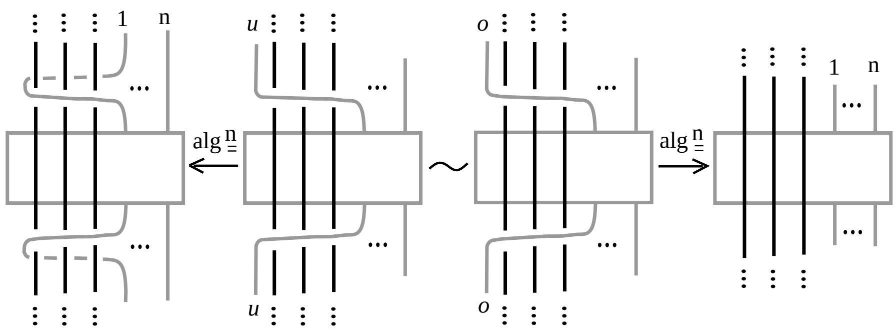

Figure 22: Conjugation by b_1 corresponds to allowed change of labels

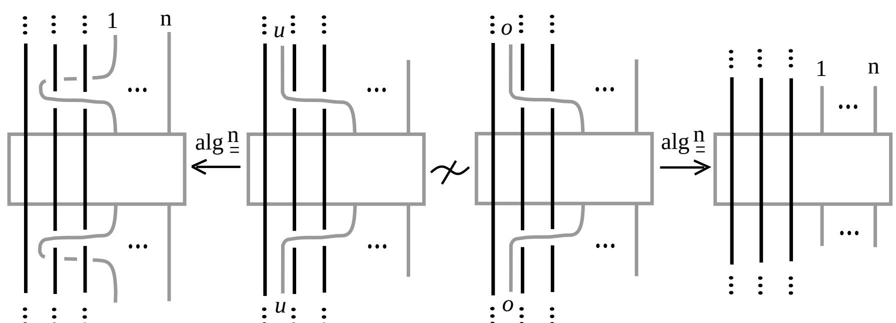

Figure 23: Conjugation by b_2 corresponds to non–allowed change of labels

as above, change of labels corresponds to conjugating the 'o' braid (resp. the 'u' braid) by the loop b_{i+1} (resp. ${b_{i+1}}^{-1}$), see Fig. 23. And conversely, two algebraic mixed braids that are conjugate by a b_i, with $i \neq 1$, can be seen as the algebraizations of two geometric mixed braids which differ only by the labels of one pair of corresponding endpoints. But such a change of labels does not reflect isotopy. To see this, think of the infinitely extended strands of I_g joining at the point at infinity. Then, the closing arc of the geometric 'u' braid would have to cross the point at infinity in order to come to the 'o' position. Consequently conjugations by the b_i's for $i \neq 1$ are not allowed, except for some obvious special cases of disconnected diagrams, which would then imply that the knot can also live in a handlebody of smaller genus.

Finally, the answer to the third question lies in the following result.

Theorem 6. *Conjugation by a maximal loop ω is the only conjugation by words* *in the b_i's, which preserves the isotopy class in H_g of the closure of* any *braid in $B_{g,n}$. That is, for any word $\beta \in B_{g,n}$ in the b_i's, different from ${b_1}^n$, for $n \in \mathbb{Z}$, there is an $\alpha \in B_{g,n}$, such that $\alpha\,\beta$ and $\beta\,\alpha$ have non-isotopic closures.*

Proof. Counter-examples of the required kind exist in the one-strand braid group $B_{g,1}$. Indeed, assume the theorem were false, i.e. the closures of $\beta\,\alpha\,\beta^{-1} \in B_{g,1}$ and $\alpha \in B_{g,1}$ are isotopic, for all α, $\beta \in B_{g,1}$ with α a word in the b_i's and β some b_r. From Theorem 4 we know that these two braids are related by braid isotopies and algebraic L-moves. Since $B_{g,1}$ is the group generated by the b_i's, and this is a free group, β cannot be commuted through α. Hence we have to invoke the L-moves. These introduce some σ_i's but they do not change the order of the b_i's. According to the relations in the braid group, this can only be done if the condition $r \leq i$ is satisfied. But this is true always, only if $r = 1$. Therefore, β has to be b_1 or its inverse. Thus, conjugation by b_r for $r \geq 2$ cannot be realized in the generic case. Topologically, this corresponds precisely to crossing the point at infinity discussed above. □

It is crucial for the whole study of braids in a handlebody to note that not all conjugations in the groups $B_{g,n}$ preserve the isotopy class of the closure of an algebraic mixed braid.

Corollary 1. *Theorem 6 disproves Conjecture 4.4 in [18].*

6 Markov functionals

Theorem 5 opens up the possibility to define invariants of links in the handlebody by algebraic considerations. This runs largely in parallel with the derivation of link invariants in S^3 from Markov traces, see for example [7]. In the handlebody case, however, trace functionals are not appropriate because not all conjugations are allowed, see Theorem 6. Hence, we have to modify the definitions, so as to take this into account.

For an integral domain R let $RB_{g,n}$ denote the group ring of the handlebody braid group. A *Markov functional* is an R-linear map

$$\mu : RB_{g,n} \longrightarrow R,$$

for which units $x, \lambda \in R^*$ exist such that:

$$\begin{aligned}
\mu(1) &= 1 && (1)\\
x\mu(\beta) &= \mu(\iota(\beta)) && (2)\\
\mu(\beta\sigma_i^{\pm 1}) &= \mu(\sigma_i^{\pm 1}\beta), \qquad \beta\sigma_i \in B_{g,n} && (3)\\
\mu(\beta_1\sigma_n^{\pm 1}\beta_2) &= \lambda^{\pm 1}\mu(\beta_1\beta_2) \qquad \beta_1, \beta_2 \in B_{g,n} && (4)
\end{aligned}$$

Here ι stands for the morphism that embeds $B_{g,n}$ into $B_{g,n+1}$ by adding an unlinked strand on the right. Moreover, we need the exponent sum of ordinary crossings

$$e : B_{g,n} \longrightarrow \mathbb{Z}, \qquad \sigma_i \mapsto 1,\ a_r \mapsto 0,\ e(\beta_1\beta_2) = e(\beta_1) + e(\beta_2).$$

Theorem 5 now implies the following:

Definition 11. The expression defined by

$$\mathcal{L}(L) := x^{n-1}\lambda^{-e(B(L))}\mu(B(L)),$$

where $B(L) \in B_{g,n}$ is an invariant of oriented links in the handlebody.

Remark 7. Nice quotients of the group algebra of $B_{g,n}$ that support Markov functionals are to be studied in further work.

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